

THE GOLOD PROPERTY FOR PRODUCTS AND HIGH SYMBOLIC POWERS OF MONOMIAL IDEALS

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ABSTRACT. We show that for any two proper monomial ideals I and J in the polynomial ring $S = k[x_1, \dots, x_n]$ the ring S/IJ is Golod. We also show that if I is squarefree then for large enough k the quotient $S/I^{(k)}$ of S by the k^{th} symbolic power of I is Golod. As an application we prove that the multiplication on the cohomology algebra of some classes of moment-angle complexes is trivial.

1. INTRODUCTION

For a graded ideal I in the polynomial ring $S = \mathbb{K}[x_1, \dots, x_n]$ in n variables over the field \mathbb{K} the ring S/I is called *Golod* if all Massey operations on the Koszul complex of S/I with respect of $\mathbf{x} = x_1, \dots, x_n$ vanish. The naming gives credit to Golod [11] who showed that the vanishing of the Massey operations is equivalent to the equality case in the following coefficientwise inequality of power-series which was first derived by Serre:

$$\sum_{i \geq 0} \dim_{\mathbb{K}} \text{Tor}_i^{S/I}(\mathbb{K}, \mathbb{K}) t^i \leq \frac{(1+t)^n}{1 - t \sum_{i \geq 0} \dim_{\mathbb{K}} \text{Tor}_i^S(S/I, \mathbb{K}) t^i}$$

We refer the reader to [1] and [8] for further information on the Golod property and to [4] and [12] for the basic concepts from commutative algebra underlying this paper. We prove the following two results.

Theorem 1.1. *Let I, J be two monomial ideals in S different from S . Then S/IJ is Golod.*

In the statement of the second result we write $I^{(k)}$ for the k^{th} symbolic power of the ideal I .

Theorem 1.2. *Let I be a squarefree monomial ideal in S different from S . Then for $k \gg 0$ the k^{th} symbolic power $I^{(k)}$ is Golod for $k \gg 0$.*

Besides the strong algebraic implications of Golodness the case of squarefree monomial ideals relates to interesting topology. Let Δ be a simplicial complex on ground set $[n]$ and let $\mathbb{K}[\Delta]$ be its Stanley–Reisner ring (see § 4 for basic facts about Stanley–Reisner rings).

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By work of Buchstaber and Panov [5, Thm. 7.7], extending an additive isomorphism from [10], it is known that there is an algebra isomorphism of the Koszul homology $H_*(\mathbf{x}, k[\Delta])$ and the singular cohomology ring $H^*(M_\Delta; k)$ where $M_\Delta = \{(v_1, \dots, v_n) \in (D^2)^n \mid \{i \mid v_i \notin S^1\} \in \Delta\}$. Here $D^2 = \{v \in \mathbb{R}^2 \mid \|v\| \leq 1\}$ is the unit disk in \mathbb{R}^2 and S^1 its bounding unit circle. Note that the isomorphism is not graded for the usual grading of $H_*(\mathbf{x}, k[\Delta])$ and $H^*(M_\Delta; k)$. The complex M_Δ is the *moment-angle complex* or *polyhedral product* of the pair (D^2, S^1) for Δ (we refer the reader to [5] and [7] for background information). Last we write $\Delta^\circ = \{A \subseteq [n] \mid [n] \setminus A \notin \Delta\}$ for the *Alexander dual* of the simplicial complex Δ . Now we are in position to formulate the following consequence of Theorem 1.1.

Corollary 1.3. *Let Δ be a simplicial complex such that $\Delta = (\Delta_1^\circ * \Delta_2^\circ)^\circ$ for two simplicial complexes Δ_1, Δ_2 on disjoint ground sets. Then the multiplication on $H^*(M_\Delta; k)$ is trivial.*

The main tool for the proof of Theorem 1.1 and Theorem 1.2 is combinatorial. Let I be a monomial ideal and write $G(I)$ for the set of minimal monomial generators of I . In [16, Def. 3.8] the author introduces a combinatorial condition on $G(I)$ that in [3] was shown to imply the Golod property for S/I . The ideal I is said to satisfy the *strong gcd-condition* if there exists a linear order \prec on $G(I)$ such that for any two monomials $u \prec v \in G(I)$ with $\gcd(u, v) = 1$ there exists a monomial $w \in G(I)$ with $w \neq u, v$ such that $u \prec w$ and w divides $\text{lcm}(u, v) = uv$. The following result from [3] removes an unnecessary assumption from the statement of Theorem 7.5 in [16].

Theorem 1.4 (Thm. 5.5 [3]). *Let I be a monomial ideal. If I satisfies the strong gcd-condition then S/I is Golod.*

We refer the reader to [2] for the relation of the gcd-condition to standard combinatorial properties of simplicial complexes.

The paper is organized as follows. In § 2 we verify the strong gcd-condition for products of monomial ideals and in § 3 for high symbolic powers of squarefree monomial ideals. This yields Theorem 1.1 and Theorem 1.2. In § 4 we study the implications of Theorem 1.1 and Theorem 1.2 on moment-angle complexes. In particular, we derive Corollary 1.3.

2. PRODUCT OF MONOMIAL IDEALS

Since by Theorem 1.4 for any monomial ideal I satisfying the strong gcd-condition the ring S/I is Golod, the following result immediately implies Theorem 1.1.

Proposition 2.1. *For any two monomial ideals I, J in S that are different from S the ideal IJ satisfies strong-gcd condition.*

Proof. Fix a monomial order $<$ on the set of monomials of S . We define a linear order \prec on $G(IJ)$ as follows: For two monomials $u, v \in G(IJ)$, we set $u \prec v$ if and only if $\deg(u) > \deg(v)$ or $\deg(u) = \deg(v)$ and $u < v$. Now the following claim states that $G(IJ)$ ordered by \prec satisfies the strong gcd-condition.

Claim: For any two monomials $u, v \in G(IJ)$ with $u \prec v$ and $\gcd(u, v) = 1$ there exists a monomial $w \in G(IJ)$ different from u and v such that $u \prec w$ and w divides $\text{lcm}(u, v) = uv$.

Since $u, v \in G(IJ)$, there exist monomials $u_1, v_1 \in G(I)$ and $u_2, v_2 \in G(J)$ such that $u = u_1u_2$ and $v = v_1v_2$. We consider two cases:

Case 1. $\deg(u) = \deg(v)$.

By definition, $u < v$. Therefore, $u_1 < v_1$ or $u_2 < v_2$, since otherwise $u = u_1u_2 \geq v_1v_2 = v$, which is a contradiction. Without loss of generality we may assume that $u_1 < v_1$.

If $\deg(u_1) \geq \deg(v_1)$, then define $w' := v_1u_2$. Now $w' \in IJ$, $\deg(w') \leq \deg(u)$ and $u < w'$. If $w' \in G(IJ)$, then we set $w = w'$ and clearly $u \prec w$ and w divides $\text{lcm}(u, v) = uv$. By $\gcd(u, v) = 1$, it follows that $u_2 \neq v_2$ and therefore $w \neq v$. If $w' \notin G(IJ)$, then there exists a monomial $w \in G(IJ)$, such that $w|w'$. Now w divides $\text{lcm}(u, v) = uv$ and since $\deg(w) < \deg(w') \leq \deg(u) = \deg(v)$, we conclude that $u \prec v \prec w$ and $w \neq v$.

If $\deg(u_1) < \deg(v_1)$, then $\deg(u_2) > \deg(v_2)$, since $\deg(u) = \deg(v)$. We define $w' := u_1v_2$. Now $w' \in IJ$ and therefore there exists a monomial, say $w \in G(IJ)$, such that $w|w'$. Then w divides $\text{lcm}(u, v) = uv$ and since $\deg(w) \leq \deg(w') < \deg(u) = \deg(v)$, we conclude that $u \prec v \prec w$ and $w \neq v$.

Case 2. $\deg(u) \neq \deg(v)$.

By definition of \prec we must have $\deg(u) > \deg(v)$. Hence either of $\deg(v_1) < \deg(u_1)$ or $\deg(v_2) < \deg(u_2)$ holds. Without loss of generality we assume that $\deg(v_1) < \deg(u_1)$. We define $w' = v_1u_2$. Then $w' \in IJ$ and therefore there exists a monomial, say $w \in G(IJ)$, such that w divides w' . Thus w divides $\text{lcm}(u, v) = uv$ and $u \prec w$, since $\deg(w) \leq \deg(w') < \deg(u)$. Also $w \neq v$, because otherwise v divides v_1u_2 and since $\gcd(u_1, v) = 1$, it implies that v divides v_1 , which is impossible. \square

In [15, Thm. 4.1] it is shown that for a homogeneous ideal $I \neq S$ in the polynomial ring S the ring S/I^k is Golod for $k \gg 0$. For monomial ideals Theorem 1.1 allows a more precise statement of their result which follows immediately from Proposition 2.1 and Theorem 1.4.

Corollary 2.2. *Let I be a monomial ideal in the polynomial ring S different from S . Then for $k \geq 2$ the ideal I^k satisfies the strong gcd-condition and hence the ring $R = S/I^k$ is Golod.*

Since by [4, Thm. 3.4.5] the Koszul homology $H_*(\mathbf{x}, S/I)$ for a Gorenstein quotient S/I of S is a Poincaré duality algebra, S/I cannot be Golod unless I is a principal ideal. Thus Theorem 1.1 implies that a product of monomial ideal can be Gorenstein if and only if it is principal. This observation is not new and can be seen as a consequence of a very general result by Huneke [14] who showed that in any unramified regular local ring no Gorenstein ideal of height ≥ 2 can be a product.

3. SYMBOLIC POWERS OF SQUAREFREE MONOMIAL IDEALS

For our results on symbolic powers we have to restrict ourselves to squarefree monomial ideals I . This is due to the fact that our proofs use that only in this case in the primary decomposition $I = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_r$ of I every \mathfrak{p}_i is an ideal of S generated by a subset of the variables of S [12, Lem. 1.5.4]. Moreover, in this situation for a positive integer k the k^{th} symbolic power $I^{(k)}$ of I coincides with $\mathfrak{p}_1^k \cap \dots \cap \mathfrak{p}_r^k$ [12, Prop. 1.4.4].

Now we are ready to prove that the high symbolic powers of squarefree monomial ideals of S fulfill the strong-gcd condition.

Proposition 3.1. *Let I be a squarefree monomial ideal in S different from S . Then for $k \gg 0$ the k^{th} symbolic power $I^{(k)}$ satisfies the strong gcd-condition.*

Proof. By [17, Prop. 1] the ring $A = \bigoplus_{i=0}^{\infty} I^{(i)}$ is Noetherian and therefore is a finitely generated \mathbb{K} -algebra. Assume that the set $\{y_1, \dots, y_m\}$ is a set of generators for the \mathbb{K} -algebra A . Following [6], since A is finitely generated, there exists a natural number c such that $A_0 = \bigoplus_{i=0}^{\infty} I^{(ci)}$ is a standard \mathbb{K} -algebra and therefore a Noetherian ring. Note that the set

$$\{y_1^{\ell_1} \dots y_m^{\ell_m} \mid 0 \leq \ell_1, \dots, \ell_m \leq c-1\}$$

is a system of generators for A as an A_0 -module. Assume that the degree of a generator of the A_0 -module A is at most α . Since A_0 is standard \mathbb{K} -algebra for every integer $k > \max\{\alpha, c\}$ we have $I^{(k)} = I^{(c)}I^{(k-c)}$. Thus for every $k > \max\{\alpha, c\}$, the ideal $I^{(k)}$ is the product of two monomial ideals. Therefore, by Proposition 2.1 it satisfies the strong-gcd condition. \square

The following corollary is an immediate consequence of Theorem 1.4 and Theorem 1.2.

Corollary 3.2. *Let I be a squarefree monomial ideal in the polynomial ring S , which is not a principal ideal. Then for $k \gg 0$ the ring $S/I^{(k)}$ is not Gorenstein.*

We do not know which k suffices. Indeed, we do not have an example of a monomial ideal $I \neq S$ squarefree or not for which $I^{(2)}$ is not Golod.

4. MOMENT-ANGLE COMPLEXES

First, we recall some basics from the theory of Stanley-Reisner ideals. Let Δ be a simplicial complex on ground set $[n]$. A subset $N \subseteq [n]$ such that $N \notin \Delta$ and $N \setminus \{i\} \in \Delta$ for all $i \in N$ is called a *minimal non-face* of Δ . The *Stanley-Reisner ideal* I_Δ of Δ is the ideal in S generated by the monomial x_N for the minimal non-faces N of Δ and the quotient $k[\Delta] = S/I_\Delta$ is Stanley-Reisner ring of Δ . Indeed, the map sending Δ to I_Δ is a bijection between the simplicial complexes on ground set $[n]$ and the squarefree monomial ideals in S . To any monomial ideal I its *polarization* I^{pol} [12, p. 19] is a squarefree monomial ideal and it is known [9, Thm. 3.5] that I is Golod if and only if I^{pol} is Golod. In addition, it follows from the vanishing of the Massey operations that the multiplication on Koszul homology $H_*(\mathbf{x}, S/I)$ is trivial for all Golod S/I – by trivial we here mean that products of two elements of positive degree are 0. Thus the algebra isomorphism of $H_*(\mathbf{x}, k[\Delta])$ and $H^*(M_\Delta; k)$ [5, Thm. 7.7] together with Theorem 1.1 and Theorem 1.2 yields the following corollary. Note that even though the isomorphism of $H_*(\mathbf{x}, k[\Delta])$ and $H^*(M_\Delta; k)$ is not graded in the usual grading it sends $H_0(\mathbf{x}, k[\Delta])$ and $H^0(M_\Delta; k)$.

Corollary 4.1. *Let I and J be monomial ideals. Then for the simplicial complexes Γ and $\Gamma^{(k)}$ such that $(IJ)^{\text{pol}} = I_\Gamma$ and $(I^{(k)})^{\text{pol}} = I_{\Gamma^{(k)}}$ we have:*

- (i) *The multiplication on the cohomology algebra of M_Γ is trivial.*

(ii) *The multiplication on the cohomology algebra of $M_{\Gamma^{(k)}}$ is trivial for $k \gg 0$.*

In general, the combinatorics and geometry of the simplicial complexes Γ and $\Gamma^{(k)}$ cannot be easily controlled even in the case I and J are squarefree monomial ideals. Therefore, the result is more useful in a situation when the ideals IJ and $I^{(k)}$ themselves are squarefree monomial ideals. Since this never happens for $I^{(k)}$ and $k \geq 2$ we confine ourselves to the case of products IJ . The following lemma shows that in this case I and J should be squarefree monomial ideals with generators in disjoint sets of variables. Even though the lemma must be a known basic fact from the theory of the monomial ideals we did not find a reference and hence for the sake of completeness we provide a proof. In the proof we denote for a monomial u by $\text{supp}(u)$ its *support*, which is the set of variables dividing u .

Lemma 4.2. *Let I, J be monomial ideals. Then IJ is a squarefree monomial ideal if and only if I and J are squarefree monomial ideals such that $\gcd(u, v) = 1$ for all $u \in G(I)$ and $v \in G(J)$.*

Proof. The “if” part of the lemma is trivial. The other direction states that if IJ is a squarefree monomial ideal then for every $u \in G(I)$ and every $v \in G(J)$ the monomial uv is squarefree. Assume by contradiction that there exist monomials $u \in G(I)$ and $v \in G(J)$ such that uv is not squarefree. Among every v with this property we choose one such that the set $\text{supp}(v) \setminus \text{supp}(u)$ has minimal cardinality. Since IJ is a squarefree monomial ideal, there exist squarefree monomials $u' \in G(I)$ and $v' \in G(J)$ such that $\text{supp}(u') \cap \text{supp}(v') = \emptyset$ and $u'v'$ divides uv . We distinguish two cases:

Case: $\text{supp}(v) \setminus \text{supp}(u) = \emptyset$

The assumption is equivalent to $\text{supp}(v) \subseteq \text{supp}(u)$. We conclude that $\text{supp}(u') \subseteq \text{supp}(uv) = \text{supp}(u)$. But then u' divides u and hence $u = u'$. But then $\text{supp}(uv) = \text{supp}(u')$ and $v' = 1$. This implies $J = S$ and $IJ = I$ which in turn shows that I is a squarefree monomial ideal. Thus u is squarefree and $v = v' = 1$. But then uv is squarefree and we arrive at a contradiction.

Case: $\text{supp}(v) \setminus \text{supp}(u) = \{x_{i_1}, \dots, x_{i_t}\}$ for some $t \geq 1$

If $\text{supp}(v') \subseteq \text{supp}(v)$ then v' divides v and hence $v = v'$. Then u' must divide u and hence $u = u'$. But this contradicts the fact that $uv = u'v'$ is not squarefree. Hence $\text{supp}(u) \cap \text{supp}(v') \neq \emptyset$ and therefore uv' is not squarefree. Now assume that $\{x_{i_1}, \dots, x_{i_t}\} \subseteq \text{supp}(v')$. Then since $\text{supp}(u') \cap \text{supp}(v') = \emptyset$, it follows that $\text{supp}(u') \subseteq \text{supp}(u)$ and therefore u' divides u . As above this yields a contradiction. Thus $\{x_{i_1}, \dots, x_{i_t}\} \not\subseteq \text{supp}(v')$ and the inclusion $\text{supp}(v') \subseteq \text{supp}(u) \cup \text{supp}(v)$ implies that the cardinality of $\text{supp}(v') \setminus \text{supp}(u)$ is strictly less than the cardinality of $\text{supp}(v) \setminus \text{supp}(u)$, which contradicts the choice of v . \square

Let us analyze the situation when IJ is a squarefree monomial ideal more carefully. By Lemma 4.2, we may assume that $I = I_{\Delta}$ and $J = I_{\Delta'}$ with simplicial complexes Δ and Δ' that have a join decomposition $\Delta = 2^{V_1} * \Delta_1$ and $\Delta' = 2^{V_2} * \Delta_2$ for simplicial complexes Δ_1 and Δ_2 over disjoint ground sets and full simplices 2^{V_1} and 2^{V_2} over arbitrary finite sets V_1 and V_2 . The product of the corresponding Stanley–Reisner ideals is easily described.

Lemma 4.3. *Let Δ_i , $i = 1, 2$, be simplicial complexes on disjoint ground sets Ω_1 and Ω_2 . Then $I_{\Delta_1} I_{\Delta_2} = I_{(\Delta_1^\circ * \Delta_2^\circ)^\circ}$.*

Proof. Recall that by definition of the Alexander dual, for every simplicial complex Δ on the ground set Ω , the maximal faces $A \in \Delta^\circ$ are those subsets A of Ω for which $\Omega \setminus A$ is a minimal non-face of Δ .

The product $I_{\Delta_1} I_{\Delta_2}$ is generated by $x_{N_1} x_{N_2}$ for minimal non-faces N_1 and N_2 of Δ_1 and Δ_2 respectively. Hence it is generated by monomials corresponding to the union of $F_1 = \Omega_1 \setminus N_1$ and $F_2 = \Omega_2 \setminus N_2$ of maximal faces of Δ_1° and Δ_2° . Since for any maximal face F of $\Delta_1^\circ * \Delta_2^\circ$ there are maximal faces F_1 of Δ_1° and F_2 of Δ_2° such that $F = F_1 \cup F_2$ it follows that the monomials $x_{N_1 \cup N_2} = x_{N_1} x_{N_2}$ for minimal non-faces N_1 of Δ_1 and N_2 of Δ_2 are the generators $I_{(\Delta_1^\circ * \Delta_2^\circ)^\circ}$. Now the assertion follows. \square

Now combining Lemma 4.3 and Corollary 4.1 (i) yields Corollary 1.3.

We note, that for the deduction of Corollary 1.3 one could have also argued using [13, Satz 2] instead of Theorem 1.1.

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